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COMMENT

On the symmetries of the Julia sets for the process $z \Rightarrow z^p + c$

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Abstract. The self-replicating properties of the Julia sets $J_c(p)$ for the iterative process $z \Rightarrow z^p + c$ are examined for integers $p > 1$. It is shown that the corresponding Mandelbrot sets $M(p)$ contain $(p-1)$-fold symmetries, which results in $J_c(p)$ having p-fold symmetries.

The Julia sets and the Mandelbrot set for the iterative process $z \Rightarrow z^2 + c$, $c \in \mathbb{C}$, $z \in \mathbb{C}$, have been widely investigated in recent years, so much so that they are the exclusive subjects of a recent book (Peitgen and Richter 1986). Briefly stated, these examinations check if the process

$$z_{n+1} = (z_n)^2 + c \tag{1}$$

remains bounded. In (1), if z_0 varies in the field $\mathbb{C}$ of complex numbers and c is fixed, a Julia set J_c results; and if $z_0 = 0$ but c varies in $\mathbb{C}$ then the Mandelbrot set M is formed.

With reference to the Julia sets of the process (1), if z_0 is far from zero, then the sequence $\{z_n\}$ converges to ∞ very quickly; and, even if some z_m turns out to be far from zero, then too the sequence $\{z_n\}$ converges to zero as $n (n > m)$ distances itself from m (Douady 1986). However, if z_0 is close to zero, then the sequence $\{z_n\}$ converges to some finite value, called a *strange attractor.* These last classes of z_0 may form simply connected *basins of attraction*, each basin containing a strange attractor. There may be one or many basins, or even none; whether or not there will be basins of attraction depends on the specific value of the complex constant c. Although the actual Julia set is defined as the boundary of the (multiply) connected set of z_0 for which the sequence $\{z_n\}$ is bounded, we will inaccurately refer to the whole z plane, suitably encoded, as the Julia set J_c. This encoding depends on the integer N: if the sequence $\{z_n\}$ is unbounded, then $|z_N|^2 > R$ and $|z_{N-1}|^2 \leq R$, where R is some arbitrarily fixed large number and N too is some large integer. Figure 1 shows the FORTRAN algorithm which can be used for this purpose.

The Mandelbrot set M is also generated by the process (1), but with $z_0 = 0$ and c varying. The boundary of the multiply connected set of c in which the sequence $\{z_n\}$ converges to some finite value is the set M, and provided $c \in M'$, the interior of M, the corresponding Julia set J_c will contain basins of attraction. Should c be otherwise, then J_c is not connected.

An important point to note is that while M is not self-similar, J_c is (Douady 1986). Furthermore, the J_c exhibit periodicity, because for a given z_0 and c, it is possible that the sequence $\{z_0, z_1, z_2, \ldots, z_k\}$ repeats itself. It is also possible that some sequences

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```
      INTEGER P
      COMPLEX Z, Z0, C
C----------------------------------------------------------------------
C     NOTES
C     (1)GIVE |XMAX-XMIN|/|YMAX-YMIN| IN THE RATIO 1024/780 FOR GENISCO SCREEN
C     (2)GIVE NA/NB ALSO IN THE SAME RATIO
C----------------------------------------------------------------------
      CALL INITT (30)
C
      PRINT*, 'GIVE C = C1+iC2.'
      READ (5,*) C1, C2
      PRINT*, 'GIVE THE GROWTH POWER P > 1.'
      READ(5,*) P
      PRINT*, 'GIVE XMIN, XMAX, YMIN & YMAX.'
      READ(5,*) XMIN, XMAX, YMIN, YMAX
      PRINT*, 'GIVE NA & NB.'
      READ(5,*) NA, NB
C
      DELX = (XMAX - XMIN)/DFLOAT(NA - 1)
      DELY = (YMAX - YMIN)/DFLOAT(NB - 1)
      C = DCMPLX(C1,C2)
      KMAX = 15
      RMAX = 1.0E+02
C
      DO 1000 NXX = 1, NA
      NX = NXX - 1
      DO 1000 NYY = 1, NB
      NY = NYY - 1
      Z0 = DCMPLX(XMIN + NX*DELX, YMIN + NY*DELY)
      K = 0
C
25    K = K+1
      Z = C + (Z0**P)
      R = CABS(Z)**2
      IF (R.GT.RMAX) KOLOR = K
      IF (R.GT. RMAX) GO TO 50
      IF (K.EQ.KMAX) KOLOR = 0
      IF (K.EQ.KMAX) GO TO 50
      Z0 = Z
      GO TO 25
C
50    IF(MOD(KOLOR,2).NE.0) CALL PNTABS(NX,NY)
1000  CONTINUE
C
      CALL FINITT(0,767)
      STOP
      END
```

Figure 1. FORTRAN algorithm to generate Julia sets for the process $z \Rightarrow z^p + c$, $c \in \mathbb{C}$, $z \in \mathbb{C}$, $p > 1$. The PLOT10 package was used on the DEC VAX 11/730 computer along with a Genisco graphics terminal. The plot subroutine PNTABS is part of the PLOT10 package.

be *preperiodic*, i.e. they fall into a repetitive cycle after a few initial steps. For the iterative process (1), these phenomena have been extensively investigated and are summarised most lucidly by Douady (1986).

Shown in the lower-left hand corners of figures 2–5 are the Julia sets drawn for the iterative process (1) with several different values of c. The coding of these maps can be gleaned from the algorithm given in figure 1, and is as follows: if $|z_N|^2 > R = 100$, $N < 16$, then the location z_0 is coloured black if N is odd; otherwise the location of z_0 is coloured white. The evidence of periodicity in these sample maps should be carefully noted: a copy of any identifiable structure can be found by rotating the picture by π radians, and trivially so when $c \approx 0$.

What happens if the iterative process (1) were to be replaced by the similar process

$$z_{n+1} = (z_n)^p + c \tag{2}$$

where p is a positive integer greater than 1? (Of course, if $p = 2$, then the familiar Julia sets $\boldsymbol{J}_c$ and the Mandelbrot set $\boldsymbol{M}$ result.) This has also been examined in figures 2–5, where the Julia sets $\boldsymbol{J}_c(p)$ have been illustrated for $p = 2$, 3, 4 and 5. In each case, one observes the presence of a p-fold symmetry in $\boldsymbol{J}_c(p)$. Particularly in figure 5, where $-c = (1.2 + \mathrm{i}0.7)$, the p-fold symmetry has manifested itself very clearly as a p-replication

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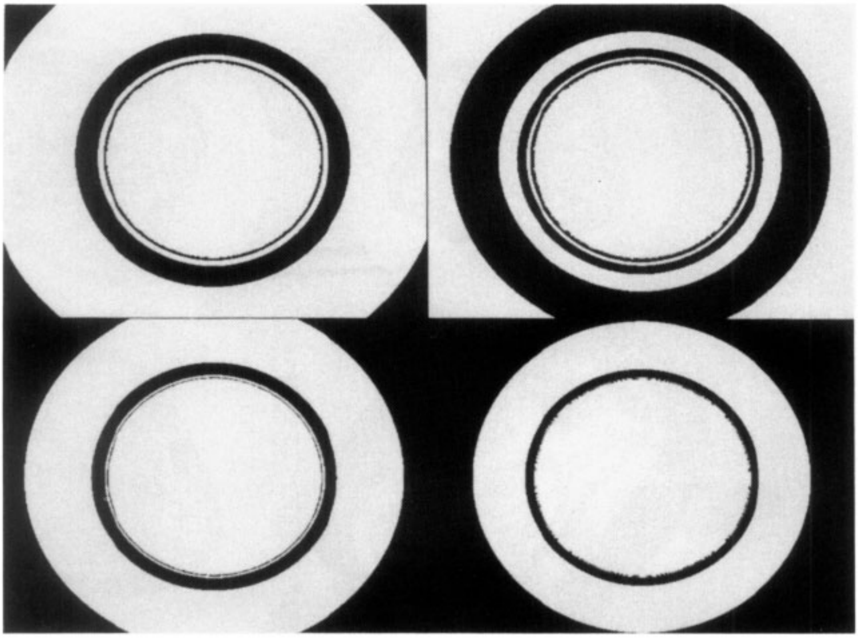

Figure 2. Julia sets $J_c(p)$ for the iterative process $z \Rightarrow z^p + c$ where $c = 0.0001 + i0.0001$. Counterclockwise, from the bottom left-hand quadrant, $p = 2, 3, 4$ and 5. For each quadrant, $|\mathrm{Re}\{z\}| \leq 2.0$, $|\mathrm{Im}\{z\}| \leq 1.6$ and the significance of black and white colouring is explained in the text.

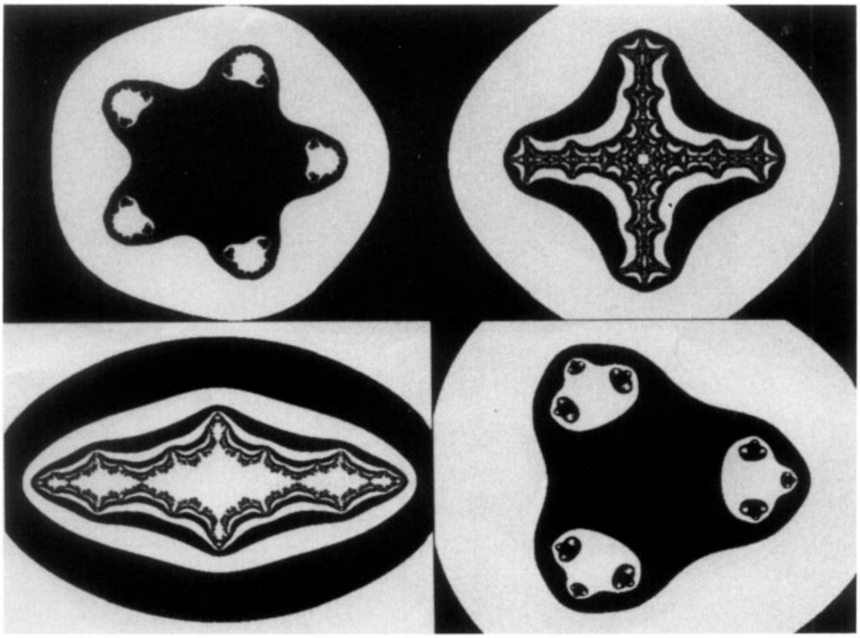

Figure 3. As for figure 2, but $c = -1.25 + i0.0$.

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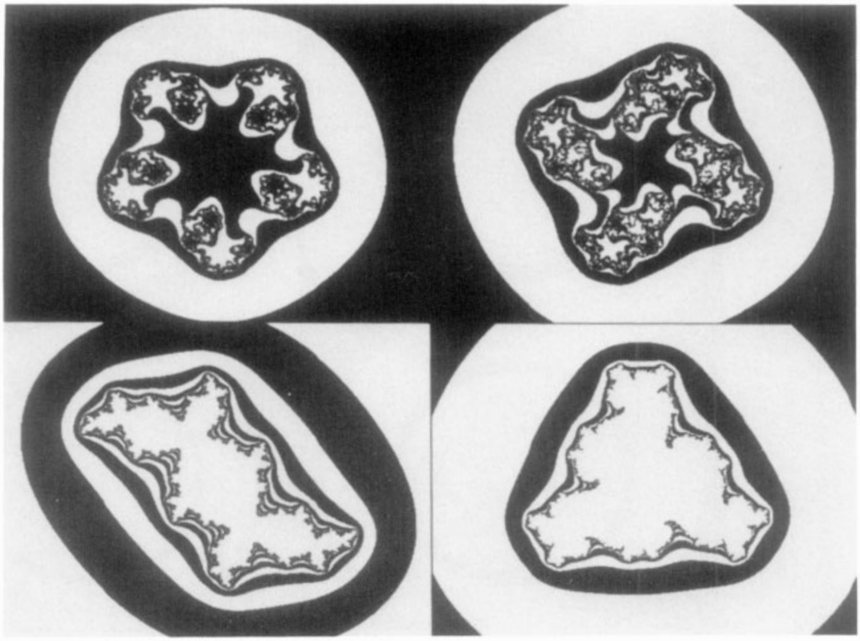

Figure 4. As for figure 2, but $c=-0.12+\mathrm{i}0.74$.

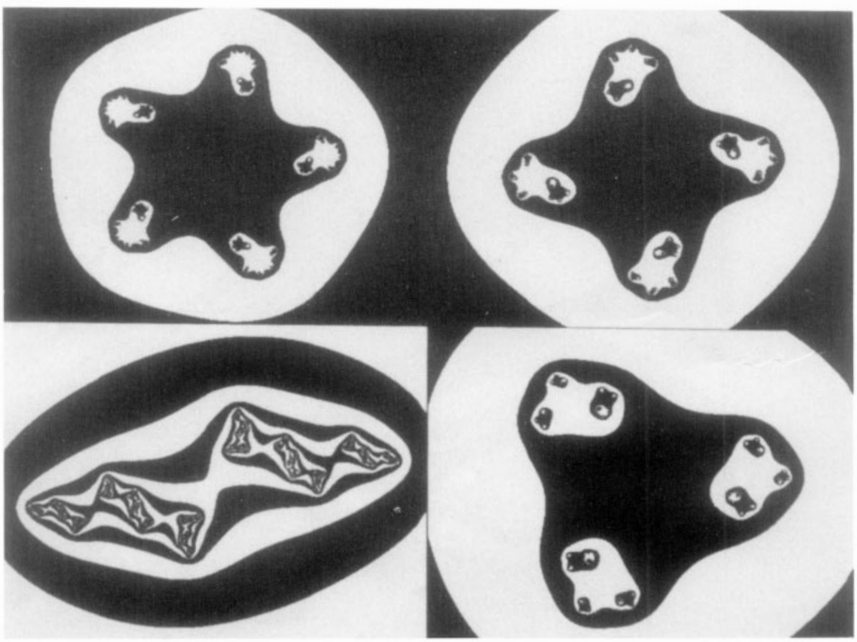

Figure 5. As for figure 2, but $c=-1.2-\mathrm{i}0.7$.

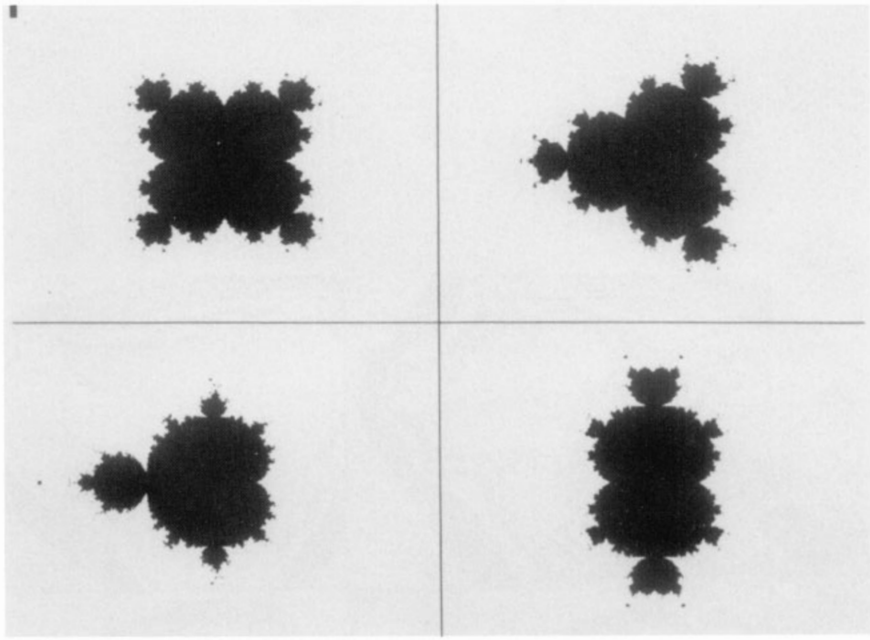

Figure 6. Mandelbrot sets $M(p)$ for the iterative process $z \Rightarrow z^p + c$. Counterclockwise, from the bottom left-hand quadrant, $p = 2, 3, 4$ and 5. For each quadrant, $|\mathrm{Re}\{c\}| \leq 2.0$, $|\mathrm{Im}\{c\}| \leq 1.6$.

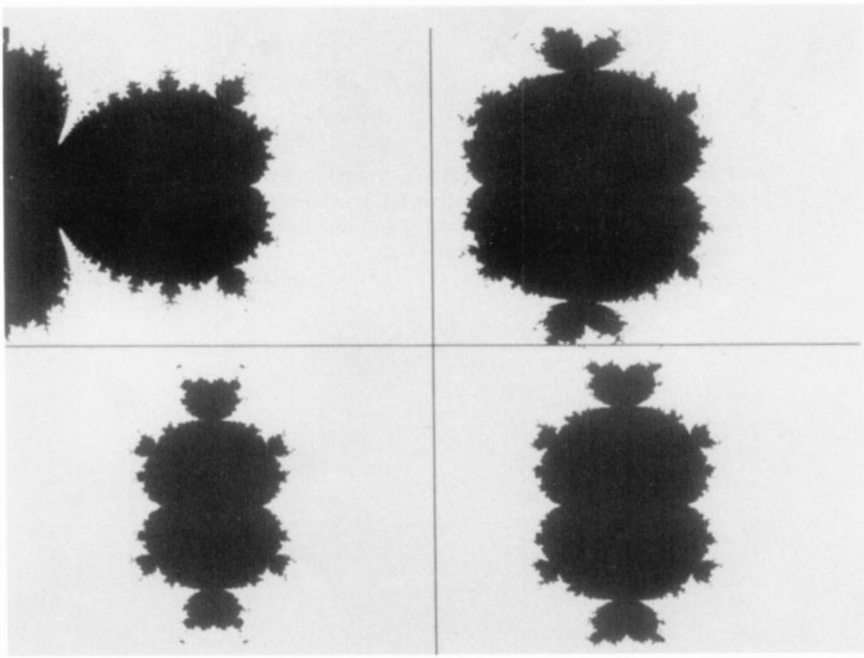

Figure 7. Mandelbrot sets for the iterative process $z \Rightarrow \alpha z^2 + (1-\alpha) z^3 + c$. Counterclockwise, from the bottom left-hand quadrant, $\alpha = 0.2$, 0.4, 0.6 and 0.8. For each quadrant, $|\mathrm{Re}\{c\}| \leq 2.0$, $|\mathrm{Im}\{c\}| \leq 1.6$.

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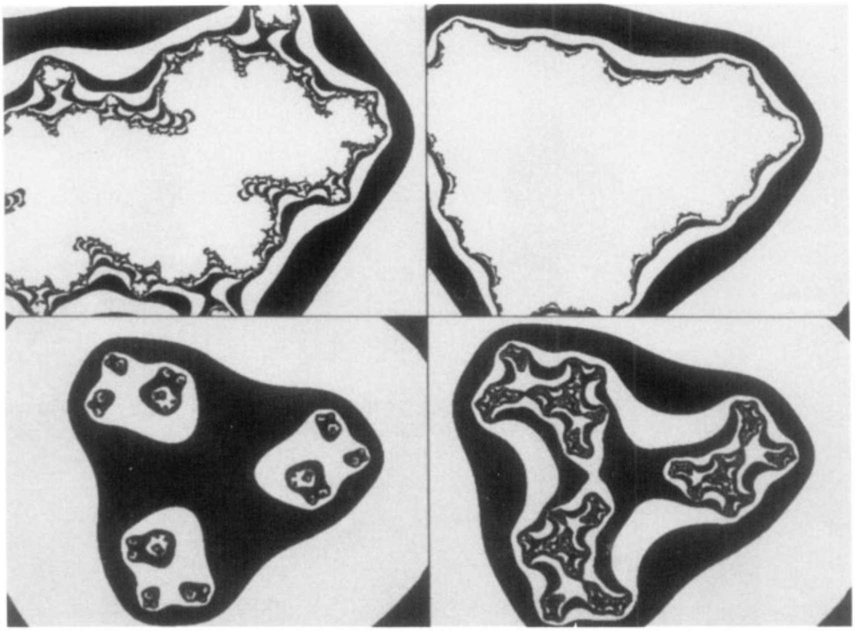

Figure 8. Julia sets for the iterative process $z \Rightarrow \alpha z^2 + (1-\alpha)z^3 + c$ where $c = -1.2 - i0.7$. Counterclockwise, from the bottom left-hand quadrant, $\alpha = 0.2, 0.4, 0.6$ and 0.8. For each quadrant, $|\mathrm{Re}\{z\}| \leq 2.0, |\mathrm{Im}\{z\}| \leq 1.6$, and the significance of black and white colouring is the same as for figures 2-5.

process, a copy of any identifiable structure can be readily obtained through the rotation of $J_c(p)$ by an angle $2\pi/p$.

Further examination of this phenomena is facilitated by figure 6, where the Mandelbrot sets $\boldsymbol{M}(p)$ corresponding to the process (2) have been plotted for $p=2$, 3, 4 and 5. It is very clear that the set $\boldsymbol{M}(p)$ exhibits a $(p-1)$-fold symmetry, the corresponding $J_c(p)$ showing a p-fold one. If c lies to the exterior of $\boldsymbol{M}(p)$, then $J_c(p)$ is unconnected and contains a fractal p-replication process.

In recent years, an interesting development has occurred in the application of the renormalisation theory for understanding phase transformations. As an example, let us consider a magnet at a temperature T which has been partitioned into N identical cubes of side a at the finest level of resolution. If, on an even coarser scale, the magnet is partitioned into identical $N'=N/b^3$ cubes of side $a'=ba$, $b>1$, it appears to have another (renormalised) temperature T'. The mapping $T'=R(T)$ is called the renormalisation transform (Wilson 1971a, b). What turns out to be of great interest is that the Julia set of the iterative process $T\Rightarrow R(T)$ for these hierarchial models is identical with the Yang–Lee set of zeros (Derrida *et al* 1983), i.e. the solutions of the polynomial equation

$$c_0+c_1z+c_2z^2+\ldots.+c_Nz^N=0. \tag{3}$$

Whether or not this identity is fortuitous, it creates a powerful incentive to explore Julia sets for polynomial iterative processes. Shown in figure 7 is the Mandelbrot set for the process $z\Rightarrow\alpha z^2+(1-\alpha)z^3+c$ for $\alpha=0.2$, 0.4, 0.6 and 0.8; in figure 8, the corresponding Julia sets are shown for a value of c which lies in the exterior of both $\boldsymbol{M}(2)$ and $\boldsymbol{M}(3)$. All the Mandelbrot sets in figure 7 are symmetric about the x axis, while all the Julia sets are symmetric about the y axis also, which means that the lower exponent of z in this mapping is the one dictating symmetry. To be noted is the fact that there is evidence of an additional symmetry in figures 7 and 8, that being due to the higher exponent of z in the mapping, but it degenerates as the value of α decreases. From these and other computations, we have concluded that the Mandelbrot set for the process

$$z\Rightarrow\left(\sum_{n=N}^{M}\alpha_n z^n\right)+c \qquad 1<N<M \tag{4}$$

exhibits a $(N-1)$-fold symmetry, the corresponding Julia sets having N symmetry axes.

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